CHAOTIC BEHAVIOUR OF CONTINUOUS DYNAMICAL SYSTEM GENERATED BY EULER EQUATION BRANCHING AND ITS APPLICATION IN MACROECONOMIC EQUILIBRIUM MODEL

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Abstract. We focus on the special type of the continuous dynamical system which is generated by Euler equation branching. Euler equation branching is a type of differential inclusion $\dot{x} \in \{f(x), g(x)\}$, where $f, g\colon X \subset \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $f(x) \neq g(x)$ at every point $x \in X$. It seems this chaotic behaviour is typical for such dynamical system.

In the second part we show an application in a new formulated overall macroeconomic equilibrium model. This new model is based on the fundamental macroeconomic aggregate equilibrium model called the IS-LM model.

Keywords: Euler equation branching; chaos; IS-LM model; QY-ML model

MSC 2010: 37N40, 91B50, 91B55

1. Introduction

In this paper we would like to briefly introduce the problem of dynamical behaviour of the continuous dynamical system generated by Euler equation branching without describing the details. We will show that there can typically exist a chaos in such systems.

The application of this problem is also interesting and we try to shortly indicate this application in macroeconomics, precisely in the new macroeconomic equilibrium model.

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2. Preliminaries

All definitions and theorems used in this section follow from [2] and are modified to the special type of differential inclusion in the plane $\mathbb{R}^2$.

Definition 2.1. Let $X \subset \mathbb{R}^2$ be an open set and let $f, g\colon X \to \mathbb{R}^2$ be continuous. Let us consider the differential inclusion given by $\dot{x} \in \{f(x), g(x)\}$. We say that there is *Euler equation branching at the point* $x \in X$ if $f(x) \neq g(x)$. If there is Euler equation branching at every point $x \in X$ then we say that there is *Euler equation branching on the set* X.

In the sequel we consider $X \subseteq \mathbb{R}^2$ is a nonempty open set with Euclidean metric d and $T := [0, \infty]$ is the time index. Let $F\colon X \to 2^{\mathbb{R}^2}$ be the set-valued function given by $F(x) := \{f(x), g(x)\}$ where $f, g\colon X \to \mathbb{R}^2$ are continuous and $f(x) \neq g(x)$ is satisfied for all $x \in X$. Let $Z = \{\gamma;\ \gamma\colon T \to X\}$, where the functions $\gamma\colon T \to X$ are continuous and continuously differentiable a.e.

Definition 2.2. The *dynamical system generated by* F is given by

$$D := \{\gamma \in Z;\ \dot{\gamma}(t) \in F(\gamma(t)) \text{ a.e.}\}.$$

Definition 2.3. We say that a nonempty $V \subset \mathbb{R}^2$ is a *compact F-invariant set*, if V is compact and for each $x \in V$ there exists a $\gamma \in D$ such that $\gamma(0) = x$ and $\gamma(t) \in V$ for all $t \in T$.

Definition 2.4. $V^* = \{\gamma \in D;\ \gamma(t) \in V \text{ for all } t \in T\}$ where $V \subset \mathbb{R}^2$ is a compact F-invariant set.

Definition 2.5. Let $a, b \in X \subseteq \mathbb{R}^2$ and let D be a dynamical system in the sense mentioned above. Let $\gamma \in D$, $t_0, t_1 \in T$ such that $t_0 < t_1$. A *simple path from a to b generated by D* is given by $P := \{\gamma(t);\ t_0 \leqslant t \leqslant t_1\}$ such that $\gamma(t_0) = a$, $\gamma(t_1) = b$ and $\dot{\gamma}$ has finitely many discontinuities on $[t_0, t_1]$ and $a \neq \gamma(s) \neq b$ for all $t_0 < s < t_1$.

Definition 2.6. Let $V \subset X \subseteq \mathbb{R}^2$ be a nonempty compact F-invariant set and $V^* = \{\gamma \in D;\ \gamma(t) \in V \text{ for all } t \in T\}$. Then V is so-called a *chaotic set* provided

(1) for all $a, b \in V$, there exists a simple path from a to b generated by V^*,

(2) there exist $U \subset V$ nonempty and open (relative to V) and $\gamma \in V^*$ such that $\gamma(t) \in V \setminus U$ for all $t \in T$ (i.e., there exists $\gamma \in V^*$ such that $\{\gamma(t);\ t \in T\}$ is not dense in V).

According to Stockman and Raines [2] chaotic sets with nonempty interior lead to the existence of chaos in the Devaney, Li-Yorke and distributional sense.

Theorem 2.1. *Let $x^* \in X \subseteq \mathbb{R}^2$, $f(x^*) = 0$ and $g(x^*) \neq 0$, let λ_1, λ_2 be eigenvalues of Jacobi's matrix of the system $\dot{x} = f(x)$ at the point x^* and let e_1, e_2 be the corresponding eigenvectors. We choose $\delta > 0$ such that $g(x) \neq 0$ for every $x \in \overline{B}_\delta(x^*)$. Let the solution of $\dot{x} = g(x)$ be not bounded in some nonempty closed subset $\overline{B}_\delta(x^*)$.*

(1) *We assume that there exists $\varepsilon > 0$ such that x^* is a source (i.e., an unstable node or focus) or a sink (i.e., a stable node or focus) for f on $B_\varepsilon(x^*)$. Then F admits a chaotic set.*
(2) *We assume that $\lambda_1 < 0$, $\lambda_2 > 0$ (i.e., x^* is a saddle point) and $g(x^*) \neq \alpha e_1$, $g(x^*) \neq \beta e_2$, where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. Then F admits a chaotic set with nonempty interior.*

3. Chaotic behaviour of dynamical system generated by Euler equation branching

In this section, we consider a continuous dynamical system generated by Euler equation branching $\dot{x} \in \{f(x), g(x)\}$. We also consider only classical singular points, which means with nonzero determinant of Jacobi's matrix, of the system considered. Further, we consider both the branches produce hyperbolic singular points and these points lie at different points in $\mathbb{R}^2$.

From Theorem 2.1 it follows that every combination of stable/unstable node/focus or saddle can produce a chaotic set. Such chaotic sets we can see in Figure 1, showing chaotic sets between a stable node and an unstable focus.

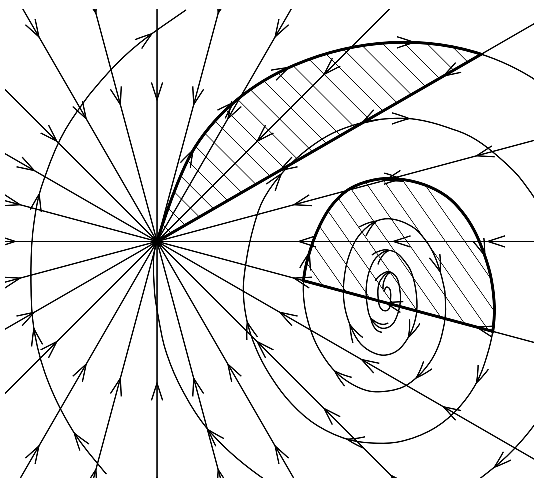

Figure 1. Chaotic sets between stable node and unstable focus.

The trajectories corresponding to the first branch (for example $\dot{x} = f(x)$) are figured by stable node type and the trajectories corresponding to the second branch (for example $\dot{x} = g(x)$) are figured by unstable focus type. The chaotic sets are displayed by hatched areas. The arrows show the directions of the trajectories. Chaotic behaviour is caused also by switching between these two branches (f and g). First the moving point goes for example along the trajectory of the stable node in the direction given by the arrow, then the moving point switches its motion to the second branch and goes along the trajectory of the unstable focus in the direction given by the arrow etc.

In Theorem 2.1 we assume that $g(x^*) \neq \alpha e_1$, $g(x^*) \neq \beta e_2$. And now we research the situation if $g(x^*) = \alpha e_1$ or $g(x^*) = \beta e_2$. We can formulate the following theorem for linear f and g, but the nonlinear case behaves exactly analogously.

Theorem 3.1. *Let $x^* \in X \subseteq \mathbb{R}^2$, $f(x^*) = 0$ and $g(x^*) \neq 0$, let λ_1, λ_2 be the eigenvalues of Jacobi's matrix of the system $\dot{x} = f(x)$ at the point x^* and e_1, e_2 the corresponding eigenvectors. We choose $\delta > 0$ such that the solution of $\dot{x} = g(x)$ is unbounded in $\overline{B}_\delta(x^*)$. We assume that $\lambda_1 < 0$, $\lambda_2 > 0$ and $g(x^*) = \alpha e_1$ or $g(x^*) = \beta e_2$, where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. Let $g(y^*) = 0$, $y^* \in X$.*

Then only if

(1) *the singular point corresponding to $\dot{x} = g(x)$ is node and $g(x^*) = \mu f(y^*)$, where $\mu > 0$,*

(2) *or the singular point corresponding to $\dot{x} = g(x)$ is saddle and $g(x^*) = -\mu f(y^*)$, where $\mu > 0$, or the other eigenvectors are collinear,*

then F does not admit chaotic set with nonempty interior.

Outline of the proof. The situation described by (1) or (2) (where F does not admit chaotic sets with nonempty interior) can be displayed as in Figure 2. The left figure corresponds to the item (1). It is a combination of the unstable node and the saddle where the unstable node lies on the stable manifold of the saddle. The same situation occurs if it is a combination of the stable node and the saddle where node lies on the unstable variety of the saddle. This follows from the condition $g(x^*) = \mu f(y^*)$. The other figures (right and in the middle) correspond to the item (2). It is the combination of two saddles. The middle figure follows from the condition $g(x^*) = -\mu f(y^*)$ and the right figure displays the situation when the other eigenvectors are collinear. So, we can see that in such situations there cannot exist the required flows (or trajectories) forming chaotic sets with nonempty interior.

Other cases admit chaotic sets with nonempty interior generated by F, see the hatched areas in Figure 3. Let $\mu > 0$. The left figure corresponds to the combination of the saddle and the node with the condition $g(x^*) = -\mu f(y^*)$ where the stable or

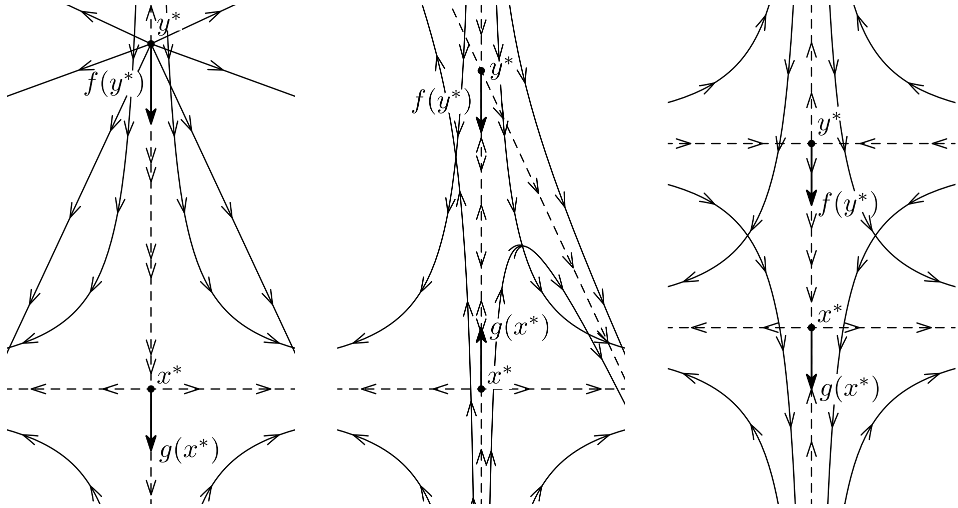

Figure 2. Schematic sketch of proof outline.

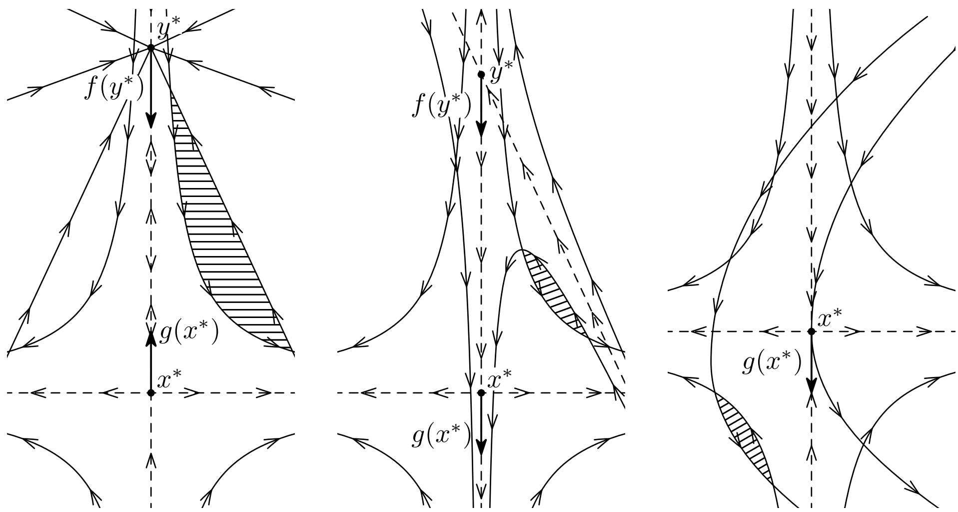

Figure 3. Schematic sketch of other cases forming chaotic sets with nonempty interior.

unstable node lies on the stable or unstable manifold. The middle figure corresponds to the combination of two saddles with the condition $g(x^*) = \mu f(y^*)$ and not collinear other eigenvectors. And the left figure displays the situation described by condition $g(x^*) \neq \mu f(y^*)$. $\square$

So, we can see that there are only few possibilities where F does not admit chaotic sets with nonempty interior. Thus, the chaotic behaviour of such dynamical systems generated by Euler equation branching is typically admitted.

4. New macroeconomic equilibrium model and its dynamical behaviour

This new macroeconomic equilibrium model describes a macroeconomic situation in two sector economy, namely the goods market equilibrium and the money market equilibrium simultaneously. The model follows from the fundamental macroeconomic model called IS-LM. The new model eliminates the deficiencies of the original model. These deficiencies are the assumptions of constant price level, of strictly exogenous money supply and of strictly demand orientation of the model. The strictly exogenous money supply means that money is a money stock determined by the central bank, and the demand orientation means that the supply is fully adapted to the demand. We newly model an inflation effect, add an endogenous view on the money supply and add a supply oriented part of the model. The endogenous money supply means that money is generated in the economy by credit creation, and the supply orientation means that the demand is fully adapted to the supply.

So, our new macroeconomic equilibrium model consists of two branches—the former is demand oriented and the latter is supply oriented. We join these two branches into one model by Euler equation branching. The switchings between these two branches are provided by the economic cycle. The demand oriented sub-model holds in the recession and the supply oriented sub-model holds in the expansion. At the peaks and troughs the models are switched.

Definition 4.1. The dynamic *overall macroeconomic IS-LM/QY-ML* model (see also [3]) is given by the differential inclusion

$$\begin{pmatrix} \dot{Y} \\ \dot{R} \end{pmatrix} \in \left\{ \begin{pmatrix} \alpha_d[I(Y,R) - S(Y,R)] \\ \beta_d[L(Y,i) - M(Y,i) - M_{CB}] \end{pmatrix}, \begin{pmatrix} \alpha_s[Q(\mathcal{K}(Y,R), \mathcal{L}(Y,R), \mathcal{T}(Y,R)) - Y] \\ \beta_s[M(Y,i) + M_{CB} - L(Y,i)] \end{pmatrix} \right\} \tag{4.1}$$

where

Y is the aggregate income (GDP, GNP),
R is the long-term real interest rate,
$i = R - MP + \pi^e$ is the short-term nominal interest rate,
$I(Y,R)$ is the investment function,
$S(Y,R)$ is the saving function,
$Q(\mathcal{K}, \mathcal{L}, \mathcal{T})$ is the production function,
$\mathcal{K}(Y,R)$ is the capital function,
$\mathcal{L}(Y,R)$ is the labour function,
$\mathcal{T}(Y,R)$ is the technical progress function,

$L(Y, i)$ is the money demand function,
$M(Y, i)$ is the money supply function,
$M_{CB} > 0$ is the money stock determined by the central bank,
$MP > 0$ is the maturity premium,
$\pi^e > 0$ is the expected inflation rate,
$\alpha_d > 0$, $\alpha_s > 0$, $\beta_d > 0$, $\beta_s > 0$ are parameters of the dynamics.

In the previous sections we considered the differential inclusion $\dot{x} \in \{f(x), g(x)\}$. Now, the separate branch $\dot{x} = f(x)$ is the modified IS-LM model:

$$\begin{aligned} \text{IS}\colon\ \dot{Y} &= \alpha_d[I(Y,R) - S(Y,R)], \\ \text{LM}\colon\ \dot{R} &= \beta_d[L(Y, R - MP + \pi^e) - M(Y, R - MP + \pi^e) - M_{CB}]. \end{aligned} \tag{4.2}$$

We can find the description of the original IS-LM model e.g. in [1]. Moreover, we distinguish two types of the interest rate (because of a modelling inflation)—the short-term nominal interest rate on the money market (equation LM) and the long-term real interest rate on the goods market (equation IS) using the well-known relation $i = R - MP + \pi^e$. Then, we furthermore add the money supply function $M(Y, i)$ as the endogenous part of the money supply. This sub-model is demand oriented. The IS-LM model describes equilibrium on the goods market and on the money market simultaneously from the demand oriented point of view. The economic equilibrium is the equality between demand and supply. The equality on the money market is obviously $L(Y, i) = M(Y, i) + M_{CB}$ in the static form. The demand side on the goods market (in two sector economy) is represented by the sum of the investment I and the consumption C and the supply side by the level of aggregate income Y (we have the demand oriented model). $S = Y - C$ holds. So, the equality of the demand and supply sides on the goods market gives $I(Y, R) = S(Y, R)$ in the static form.

The separate branch $\dot{x} = g(x)$ is a new model called the QY-ML model:

$$\begin{aligned} \text{QY}\colon\ \dot{Y} &= \alpha_s[Q(\mathcal{K}(Y,R), \mathcal{L}(Y,R), \mathcal{T}(Y,R)) - Y], \\ \text{ML}\colon\ \dot{R} &= \beta_s[M(Y, R - MP + \pi^e) + M_{CB} - L(Y, R - MP + \pi^e)]. \end{aligned} \tag{4.3}$$

This model is supply oriented. The construction proceeds as follows. On the goods market the demand side is represented by the level of the aggregate income Y (we have the supply oriented model) and the supply side is given by the aggregate production, more precisely by the aggregate production function Q. The production depends on the capital $\mathcal{K}$, labour $\mathcal{L}$ and technical progress $\mathcal{T}$. So, the equality between the goods demand and goods supply gives $Q(\mathcal{K}(Y, R), \mathcal{L}(Y, R), \mathcal{T}(Y, R)) = Y$

in the static form. The equality between the money supply and money demand on the money market is the same as in the demand oriented model, i.e., $M(Y,i) = L(Y,i)$.

Every economic function has some properties. These properties influence the dynamical behaviour of the model. The type of the hyperbolic singular point of each branch depends on the properties of the corresponding functions (i.e., I, S, L, M and Q). Any of the above mentioned combinations of the singular points may appear. The dynamical system generated by this special Euler equation branching with one branch IS-LM (sub)model and the second branch QY-ML (sub)model can typically produce a chaotic set with nonempty interior which leads to Devaney, Li-Yorke and distributional chaos.

Thus, if we identify the types of singular points in the branch represented by the IS-LM model and in the branch represented by the QY-ML model and their position in the plane $\mathbb{R}^2$ (with coordinates Y and R), we will determine whether chaotic sets with nonempty interior can or cannot in the economy arise. We can illustrate this by the following example.

Example 4.1. The economic situation can be described by Figure 4. The curve IS, curve LM and the corresponding trajectories display the dynamical behaviour of the branch $f(x)$, i.e., the demand oriented IS-LM (sub)model. The curve QY, curve ML and the corresponding trajectories display the dynamical be-

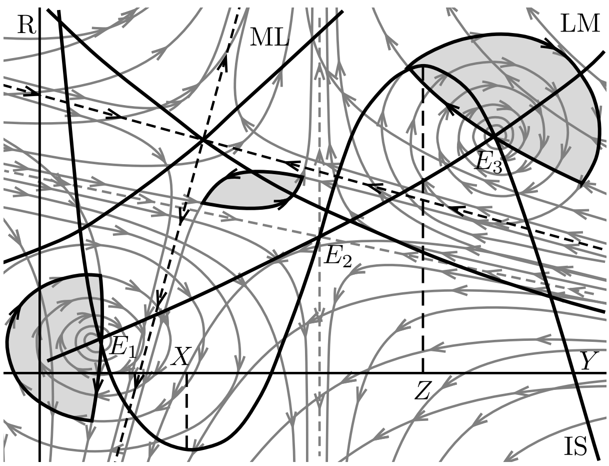

Figure 4. Example of IS-LM/QY-ML model.

haviour of the branch $g(x)$, i.e., the supply oriented QY-ML (sub)model. Such concrete shapes of these curves and types of singular points follow from the concrete economic properties of the corresponding functions. This IS-LM (sub)model has three equilibrium points E_1, E_2 and E_3. Points E_1 and E_3 are stable foci, E_2 is an unstable saddle. This QY-ML (sub)model has only one equilibrium point which is an unstable saddle point. These two branches switch according to the phase of the economic cycle. The gray-coloured areas are the chaotic sets arising in such economic system.

5. Conclusions

The chaotic behaviour of the continuous dynamical system generated by Euler equation branching in $\mathbb{R}^2$ can be typical. There are only few possibilities which cannot produce chaotic sets with nonempty interior.

Using the new overall macroeconomic equilibrium model IS-LM/QY-ML, we can show that chaotic behaviour in macroeconomics can be equally typical.

For more details, explanations and generalizations see [3].

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